TILTING MODULES OVER ALMOST PERFECT DOMAINS

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Abstract. We provide a complete classification of all *tilting modules* and *tilting classes* over almost perfect domains, which generalizes the classifications of tilting modules and tilting classes over Dedekind and 1-Gorenstein domains. Assuming the APD is Noetherian, a complete classification of all *cotilting modules* is obtained (as duals of the tilting ones).

1. Introduction

Throughout, R is a commutative ring with $1_R \neq 0_R$ and all R-modules are unital. With $Z(R)$ we denote the set of zero-divisors of R and set $R^{\times} := R\backslash Z(R)$. With $Q = (R^{\times})^{-1}R$ we denote the total ring of quotients of R (the field of quotients, if R is an integral domain). With R-Mod we denoted the category of R-modules.

Let M be an R-module. The *character module* of M is $M^c := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. With $\mathrm{Max}(M)$ we denote the (possibly empty) spectrum of maximal R-submodules and define

$$\mathrm{rad}({}_RM) := \bigcap_{L\in\mathrm{Max}(M)} L \ \ (= M, \text{ if } \mathrm{Max}(M) = \varnothing).$$

In particular, $\mathrm{Max}(R)$ is the spectrum of maximal R-ideals and $J(R) := \mathrm{rad}({}_RR)$ is the Jacobson radical of R. We denote with $\mathrm{p.d.}_R(M)$ (resp. $\mathrm{i.d.}_R(M)$, $\mathrm{w.d.}_R(M)$) the projective (resp. injective, weak or flat) dimension of ${}_RM$. Moreover, we set

$$\begin{array}{lllllll}\mathcal{P}_n & := & \{{}_RM \mid \mathrm{p.d.}_R(M) \leq n\}; & \quad & \mathcal{P} & := & \bigcup_{n=0}^{\infty} \mathcal{P}_n; \\ \mathcal{I}_n & := & \{{}_RM \mid \mathrm{i.d.}_R(M) \leq n\}; & & \mathcal{I} & := & \bigcup_{n=0}^{\infty} \mathcal{I}_n; \\ \mathcal{F}_n & := & \{{}_RM \mid \mathrm{w.d.}_R(M) \leq n\}; & & \mathcal{F} & := & \bigcup_{n=0}^{\infty} \mathcal{F}_n.\end{array}$$

In particular, $\mathcal{PR} := \mathcal{P}_0$ is the class of projective R-modules, $\mathcal{IN} := \mathcal{I}_0$ is the class of injective R-modules, and $\mathcal{FL} := \mathcal{F}_0$ is the class of flat R-modules. The class of torsion-free R-modules will be denoted with $\mathcal{TF}$. For a multiplicative subset $S \subseteq R^{\times}$, the class of S-*divisible* R-modules is

$$\mathcal{D}_S := \{{}_RM \mid sM = M \text{ for every } s \in S\}.$$

In particular, $\mathcal{DI} := \mathcal{D}_{R^{\times}}$ is the class of *divisible* R-modules. For any unexplained definitions and terminology on domains and their modules we refer to [28].

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It is well known that every module over any ring has an *injective envelope* as shown by B. Eckmann and A. Schopf [19] (see [54, 17.9]). The dual result does not hold for the categorical dual notion of *projective covers*. Rings over which every (finitely generated) module has a projective cover were considered first by H. Bass [5] and called (*semi-*)*perfect rings*. At the beginning of the current century, L. Bican, R. El Bashir, and E. Enochs [8] solved the so-called *flat cover conjecture* proving that every module has a flat cover. Recalling that the class of strongly flat modules $\mathcal{SFL}$ lies strictly between $\mathcal{FL}$ and $\mathcal{PR}$, rings over which every (finitely generated torsion) module has a *strongly flat cover* were studied by S. Bazzoni and L. Salce [13]; such rings were characterized as being *almost* (*semi-*)*perfect,* in the sense that every proper homomorphic image of such rings is (semi-)perfect (see also [14]). Since almost perfect rings that are not domains are perfect, and since perfect domains are fields, the interest is restricted to almost perfect domains (*APD*'s). Although local APD's were studied earlier by R. Smith [48] under the name "*local domains with topologically T-nilpotent radical*" (*local TTN-domains*), the interest in them resurfaced only recently in connection with the revival of theory of *cotorsion pairs* introduced by L. Salce [42]. Our main reference on APD's and their modules is the survey by L. Salce [47] (see also [13], [57], [14], [50], [44], [46], [58], [26]).

Tilting modules were introduced by S. Brenner and M. Butler [7] and then generalized by several authors (e.g. [34], [39], [18], [55], [1]). Cotilting modules appeared as vector space duals of tilting modules over finite dimensional (Artin) algebras (e.g. [33, IV.7.8.]) and then generalized in a number of papers (e.g. [17], [1], [56], [9]). A classification of (co)tilting modules over special classes of commutative rings and domains was initiated by R. Göbel and Trlifaj [30], who classified (co)tilting Abelian groups (assuming Gödel's axiom of constructibility; a condition removed later in [10]). (Co)tilting modules were classified also over Dedekind domains by S. Bazzoni et al. [10] (removing set theoretical assumptions in [53]), over valuation and Prüfer domains by L. Salce in [43] and [45], and recently over arbitrary 1-Gorenstein rings by J. Trlifaj and D. Pospíšil [52].

An open problem in [31, Page 254] is "*Characterize all tilting modules and classes over Matlis domains*" (R is Matlis, iff $\mathrm{p.d.}_R(Q) = 1$). Recalling that APD's are Matlis domains by [47, Proposition 2.5], a natural question in this connection was raised to the first author by L. Salce: "*Characterize all tilting modules and classes over APD's*". Our main result (Theorem 4.14) provides a complete answer:

MAIN THEOREM. Let R be an APD that is not a field.

(1) All tilting R-modules are 1-tilting and represented (up to equivalence) by
$$\{T(X) := \bigcap_{\mathfrak{m}\in X} R_{\mathfrak{m}} \bigoplus \frac{\bigcap_{\mathfrak{m}\in X} R_{\mathfrak{m}}}{R} \mid X \subseteq \mathrm{Max}(R)\}.$$

(2) $\{X\text{-Div} \mid X \subseteq \mathrm{Max}(R)\}$ is the class of all tilting classes, where
$$X\text{-Div} := \{{}_R M \mid \mathfrak{m}M = M \text{ for every } \mathfrak{m} \in X\}.$$

(3) If R is *coprimely packed*, then the set of *Fuchs-Salce tilting modules*
$$\{\delta_S \mid S \subseteq R^{\times} \text{ is a multiplicative subset}\}$$
classifies all tilting R-modules (up to equivalence).

This provides a partial solution to the above mentioned open problem on Matlis domains and generalizes the classification of tilting modules over 1-Gorenstein domains (which are properly contained in the class of APD's) and Dedekind domains.

The paper is organized as follows. After this introductory section, we collect in Section 2 some preliminaries on (semi-)perfect rings and almost (semi-)perfect domains. In Section 3, we characterize some classes of modules over APD's:

$$\mathcal{I}=\mathcal{I}_1, \mathcal{F}=\mathcal{F}_1=\mathcal{P}_1=\mathcal{P}, \mathcal{IN}=\mathcal{DI}\cap\mathcal{I}_1, \mathcal{FL}=\mathcal{TF}\cap\mathcal{P}_1, \mathcal{DI}=\{M \mid \mathrm{rad}({}_RM)=M\}.$$

Although these results are meant to serve in proving the main result (Theorem 4.14), we include them in a separate section since we believe they are interesting for their own. In Section 4, we present our main results. Since $\mathcal{I}=\mathcal{I}_1$ and $\mathcal{P}=\mathcal{P}_1$, we notice first that all (co)tilting modules over APD's are 1-(co)tilting. Moreover, we conclude (analogous to the case of Prüfer domains) that all torsion-free tilting modules over APD's are projective. In the local case, we prove that every tilting module over a local APD is either divisible or projective (see Theorem 4.10). Finally, we present in Theorem 4.14 a complete classification of all tilting modules over APD's that are not fields. Assuming moreover that the APD R is *coprimely packed* (e.g. R is a semilocal), we show that any tilting module is equivalent to a *Fuchs-Salce tilting R-module* δ_S for some suitable multiplicative subset $S \subseteq R^\times$. If R is a coherent (whence Noetherian) APD, then the cotilting R-modules are precisely the (dual) character modules of the tilting ones (see Corollary 4.16).

2. Preliminaries

In this section, we collect some preliminaries on (semi-)perfect rings and almost (semi-)perfect domains.

Definition 2.1. ([5]) The ring R is said to be (**semi-**)**perfect**, iff every (finitely generated) R-module has a projective cover.

For the convention of the reader, we collect in the following lemma some of the characterizations of perfect commutative rings (e.g. [2, Section 28], [54, Section 43], [36, Chapter 8], [14, Theorem 1.1]):

Lemma 2.2. *The following are equivalent:*

(1) *R is perfect;*
(2) *every semisimple R-module has a projective cover;*
(3) *every flat R-module is (self-)projective;*
(4) *direct limits of projective R-modules are (self-)projective;*
(5) *R is semilocal and every non-zero R-module has a maximal submodule;*
(6) *R is semilocal and every non-zero R-module contains a simple submodule;*
(7) *R contains no infinite set of orthogonal idempotents and every non-zero R-module contains a simple submodule;*
(8) *$R/J(R)$ is semisimple and $J(R)$ is T-nilpotent;*
(9) *$R/J(R)$ is semisimple and R is semiartinian;*
(10) *R satisfies the DCC for principal (finitely generated) ideals;*
(11) *Any R-module satisfies the DCC on its cyclic (finitely generated) R-submodules;*
(12) *Any R-module satisfies the ACC on its cyclic R-submodules;*
(13) *R is a finite direct product of local rings with T-nilpotent maximal ideals;*
(14) *R is semilocal and $R_{\mathfrak{m}}$ is a perfect ring for every $\mathfrak{m} \in \mathrm{Max}(R)$;*
(15) *R is semilocal and semiartinian.*

Definition 2.3. ([13], [14]) R is an **almost (semi-)perfect ring**, iff R/I is (semi-)perfect for every non-zero ideal $0 \neq I \trianglelefteq R$.

Remark 2.4. An almost perfect ring that is not a domain is necessarily perfect by [14, Proposition 1.3]. On the other hand, any perfect domain is a field (e.g. [47, Corollary 1.3]). This restricts the interest to *almost perfect domains* (*APD*'s).

Lemma 2.5. ([13, Theorem 4.9], [28, Theorem IV.3.7]) *The following are equivalent for an integral domain* R :

(1) *R is almost semi-perfect;*
(2) *every finitely generated torsion R-module has a strongly flat cover;*
(3) $Q/R \simeq \bigoplus_{\mathfrak{m}\in \mathrm{Max}(R)} (Q/R)_{\mathfrak{m}}$ *canonically;*
(4) *R is h-local (i.e. R/I is semilocal for every non-zero ideal $0 \neq I \trianglelefteq R$ and R/P is local for every non-zero prime ideal $0 \neq P \in \mathrm{Spec}(R)$).*

In the following lemma we collect several characterizations of APD's (see [47, Main Theorem], [13], and [14]):

Lemma 2.6. *For an integral domain R with $Q \neq R$ the following are equivalent:*

(1) *R is an APD;*
(2) *R is almost semi-perfect and $R_{\mathfrak{m}}$ is an APD for every $\mathfrak{m} \in \mathrm{Max}(R)$;*
(3) *R is h-local and $R_{\mathfrak{m}}$ is an APD for every $\mathfrak{m} \in \mathrm{Max}(R)$;*
(4) *R is h-local and Q/R is semiartinian;*
(5) *R is h-local and for every proper non-zero ideal $I \neq 0, R$, the R-module R/I contains a simple R-submodule.*
(6) *every flat R-module is strongly flat;*
(7) *every R-module has a strongly flat cover;*
(8) *every weakly cotorsion R-module is cotorsion;*
(9) *every R-module with weak dimension at most 1 has projective dimension at most 1 (i.e. $\mathcal{F}_1 = \mathcal{P}_1$);*
(10) *every divisible R-module is weak-injective.*

Remarks 2.7. Let R be an integral domain.

(1) R is a coherent APD if and only if R is Noetherian and 1-dimensional (see [13, Propositions 4.5, 4.6]). Whence, Dedekind domains are precisely the Prüfer APD's.
(2) A valuation domain R is an APD if and only if R is a DVR (e.g. [47, Example 2.2]).
(3) We have the following implications (e.g. [28], [47]): R is Dedekind $\Rightarrow$ R is 1-Gorenstein $\Rightarrow$ R is 1-dimensional and Noetherian $\Rightarrow$ R is an APD $\Rightarrow$ R is a 1-dimensional h-local $\Rightarrow$ R is a Matlis domain.

The following examples illustrate that the implications above are not reversible:

Examples 2.8. (1) Let d be a square-free integer such that $d \equiv 1 \pmod 4$ and consider the commutative Noetherian subring

$$R := \{\frac{m}{2n+1} + \frac{m'}{2n'+1}\sqrt{d} \mid m, m', n, n' \in \mathbb{Z}\} \subseteq \mathbb{Q}[\sqrt{d}].$$

By [49, Corollary 4.5], R is a 1-Gorenstein domain that is not Dedekind.

(2) Let K be a field. Then $R = K[|t^3, t^5, t^7|]$ is a Noetherian 1-dimensional domain which is not 1-Gorenstein (e.g. [38, Ex. 18.8]).

(3) Let K be a field and $V = (K[[x]], M)$ the local domain of power series in the indeterminate x with coefficients in K and with maximal ideal $M := xK[[x]]$. Let $(D, \mathfrak{m})$ be a local subring of K and consider the local integral domain $R := (D + M, \mathfrak{m} + M)$. By [14, Lemma 3.1], R is an APD if and only if D is a field. Moreover, by [14, Example 3.3], if $D = F$ is a field and $K = F(X)$, then R is Noetherian if and only if $[K : F] < \infty$. So, if $[K : F] = \infty$ then R is a *non-Noetherian* APD whence not 1-Gorenstein.
(4) Any rank-one non-discrete valuation domain is a 1-dimensional local Matlis domain that is not an APD (a concrete example is [58, Example 1.3]).
(5) Any almost Dedekind domain which is not Dedekind is a 1-dimensional Matlis domain that is not of finite character, whence not h-local (for a concrete example see [28, Example III.5.5]).

Generalizing the so-called *Prime Avoidance Theorem* (e.g. [51, 3.61]) by allowing *infinite* unions of prime ideals led to the following notions.

2.9. ([41], [21]) An ideal I of a commutative ring R is said to be *coprimely packed* (resp. *compactly packed*), iff for any set of maximal (resp. prime) R-ideals $\{P_\lambda\}_\Lambda$ we have

$$I \subseteq \bigcup_{\lambda \in \Lambda} P_\lambda \Rightarrow I \subseteq P_{\lambda_0} \text{ for some } \lambda_0 \in \Lambda. \tag{1}$$

A class of R-ideals $\mathcal{E}$ said to be *coprimely packed* (resp. *compactly packed*), iff every ideal in $\mathcal{E}$ is so. The ring R is said to be *coprimely packed* (resp. *compactly packed*), iff every ideal of R is coprimely packed (resp. compactly packed).

Remark 2.10. By [23, Lemma 2] (resp. [6, Theorem 2.3]), a ring R is coprimely packed (resp. compactly packed) if and only if $\mathrm{Spec}(R)$ is coprimely packed (resp. compactly packed). Indeed, 1-dimensional rings (e.g. APD's) are coprimely packed if and only if they are compactly packed. By [41] a Dedekind domain is compactly packed (equivalently coprimely packed) if and only if its ideal class group is torsion (see also [21, Theorem 1.4]). Semilocal rings are obviously coprimely packed (by the Prime Avoidance Theorem). A coprimely packed domain R is h-local if, for example, R is 1-dimensional by [21, Proposition 1.3] and [37, Theorem 3.22] (see also [28, Theorem 3.7, EX. IV.3.3]) or if Q/R is injective by [16, Theorem 9]. While clearly all compactly packed rings are coprimely packed, it had been shown in [41] that a Noetherian compactly packed ring has Krull dimension at most one; thus any semilocal Noetherian ring with Krull dimension at least 2 is coprimely packed but not compactly packed.

Example 2.11. Let K be an algebraically closed field and F a proper subfield such that $[K : F] = \infty$ and X an indeterminate. By [47, Example 5.5], $R := F + XK[X]$ is a non-coherent APD with $\mathrm{Max}(R) = \{XK[X]\} \cup \{(1 - aX)R \mid a \in K^\times\}$. Clearly, R is a coprimely packed (compactly packed) APD that is not semilocal.

3. Modules over APD's

In this section, we characterize the injective modules, the torsion-free modules, and the divisible modules over almost perfect domains. Moreover, we show that over such integral domains $\mathcal{I} = \mathcal{I}_1$, $\mathcal{F} = \mathcal{F}_1 = \mathcal{P}_1 = \mathcal{P}$. Throughout in this section, R is an almost perfect domain with $Q \neq R$.

Dedekind domains are characterized by the fact that every divisible module is injective (e.g. [40, Theorem 4.24], [54, 40.5]). This inspires:

Proposition 3.1. *An R-module M is injective if and only if M is divisible and* $\mathrm{i.d.}_R(M) \leq 1$, *i.e.*

$$\mathcal{IN} = \mathcal{DI} \cap \mathcal{I}_1. \tag{2}$$

Proof. ($\Rightarrow$) Injective modules over any ring are divisible (e.g. [54, 16.6]).

($\Leftarrow$) Assume that $_RM$ is divisible and $\mathrm{i.d.}_R(M) \leq 1$.

Case 1. $(R, \mathfrak{m})$ is *local.* Let $0 \neq r \in R$ be arbitrary. By Lemma 2.6 (5), the R-module R/Rr contains a simple R-submodule J/Rr ($\simeq R/\mathfrak{m}$, since $\mathrm{Max}(R) = \{\mathfrak{m}\}$). So, we have a short exact sequence of R-modules

$$0 \to J/Rr \to R/Rr \to R/J \to 0.$$

Applying the contravariant functor $\mathrm{Hom}_R(-, M)$, we get a long exact sequence

$$\cdots \to \mathrm{Ext}^1_R(R/Rr, M) \to \mathrm{Ext}^1_R(J/Rr, M) \to \mathrm{Ext}^2_R(R/J, M) \to \cdots$$

Since $_RM$ is divisible, we have $\mathrm{Ext}^1_R(R/Rr, M) = 0$ by [28, Lemma I.7.2]; and since $\mathrm{i.d.}_R(M) \leq 1$, we have $\mathrm{Ext}^2_R(R/J, M) = 0$. It follows that $\mathrm{Ext}^1_R(R/\mathfrak{m}, M) \simeq \mathrm{Ext}^1_R(J/Rr, M) = 0$, whence $_RM$ is injective by [47, Proposition 8.1. (1)].

Case 2. R is arbitrary. Let $\mathfrak{m} \in \mathrm{Max}(R)$ be arbitrary. Since R is h-local, it follows by [28, Theorem IX.7.6] that localizing any injective coresolution of R-modules at $\mathfrak{m}$ yields an injective coresolution of $R_\mathfrak{m}$-modules, hence $\mathrm{i.d.}_{R_\mathfrak{m}}(M_\mathfrak{m}) \leq 1$. Since $_{R_\mathfrak{m}}M_\mathfrak{m}$ is also divisible, we conclude that $_{R_m}M_\mathfrak{m}$ is injective by the proof of Case 1. Since R is h-local, we have (e.g. [37], [28, Theorem IX.7.6])

$$\mathrm{i.d.}_R(M) = \sup\{\mathrm{i.d.}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R)\} = 0. \blacksquare$$

It is well-known that for 1-Gorenstein domains (and general 1-Gorenstein rings), we have $\mathcal{I} = \mathcal{I}_1 = \mathcal{F} = \mathcal{F}_1 = \mathcal{P} = \mathcal{P}_1$ (e.g. [20, 9.1.10], [30, 7.1.12]). For the strictly larger class of APD's (see Example 1 (3)), these hold partially.

Proposition 3.2. *We have*

$$\mathcal{I} = \mathcal{I}_1, \ \mathcal{F} = \mathcal{F}_1 = \mathcal{P}_1 = \mathcal{P}. \tag{3}$$

Proof. Let R be an APD.

- We prove, by induction, that any R-module M with finite injective dimension at most n has injective dimension at most 1. If $n = 0$, we are done. Let $n \geq 1$ and assume the statement is true for $n - 1$. Let

$$0 \to M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \to \cdots \longrightarrow E_{n-2} \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} E_n \longrightarrow 0$$

  be an injective coresolution of $_RM$ and $L := \mathrm{Im}(f_{n-1}) = \mathrm{Ker}(f_n)$. Being a homomorphic image of a divisible R-module, L is divisible and obviously $\mathrm{i.d.}_R(L) \leq 1$ whence $_RL$ is injective by Proposition 3.1. It follows that $\mathrm{i.d.}_R(M) \leq n - 1$, whence $\mathrm{i.d.}_R(M) \leq 1$ by the induction hypothesis.
- Let M be with finite weak (flat) dimension at most n. By [28, Proposition IX. 7.7] we have for any injective cogenerator $_R\mathbf{E}$:

$$\mathrm{i.d.}_R(\mathrm{Hom}_R(M, \mathbf{E})) = \mathrm{w.d.}_R(M) \tag{4}$$

  and we conclude that $\mathrm{w.d.}_R(M) \leq 1$ by the first part of the proof.
- Let $_RM$ be with finite projective dimension at most n. Since $\mathrm{w.d.}_R(M) \leq \mathrm{p.d.}_R(M) \leq n$, we have $M \in \mathcal{F}_1 = \mathcal{P}_1$ by Lemma 2.6 (9). $\blacksquare$

Using Proposition 3.2 we conclude that an APD is either Dedekind or has (weak) global dimension ∞. This provides new characterizations of Dedekind domains and recovers the fact that Dedekind domains are precisely the Prüfer APD's.

Corollary 3.3. *An arbitrary integral domain R is Dedekind if and only if R is an APD with finite* (*weak*) *global dimension if and only if R is an APD with* (*weak*) *global dimension at most one if and only if R is a Prüfer APD.*

Proposition 3.4. *An R-module M is flat if and only if M is torsion-free and* $\mathrm{p.d.}_R(M) \leq 1$, *i.e.*

$$\mathcal{FL} = \mathcal{TF} \cap \mathcal{P}_1 = \mathcal{TF} \cap \mathcal{F}_1. \tag{5}$$

Proof. ($\Rightarrow$) Follows by the well-known fact that flat modules over domains are torsion-free (e.g. [54, 36.7]). So, we are done by $\mathcal{F}_1 = \mathcal{P}_1$ (Lemma 2.6 (9)).

($\Leftarrow$) Since ${}_RM$ is torsion-free, it embeds in a vector space over Q (e.g. [40, Lemma 4.33]). So, we have a short exact sequence of R-modules

$$0 \to M \to Q^{(\Lambda)} \to Q^{(\Lambda)}/M \to 0.$$

Since ${}_RQ^{(\Lambda)}$ is flat, $\mathrm{p.d.}_R(Q^{(\Lambda)}) \leq 1$ by Lemma 2.6 (9). It follows by [28, Lemma VI.2.4] that $\mathrm{p.d.}_R(Q^{(\Lambda)}/M) < \infty$, whence $Q^{(\Lambda)}/M \in \mathcal{P}_1 = \mathcal{F}_1$ by Proposition 3.2. Consequently, ${}_RM$ is flat.$\blacksquare$

3.5. ([31]) An R-module over an (arbitrary ring) R is said to be **strongly finitely presented**, iff it possesses a projective resolution consisting of finitely generated R-modules. With R-mod we denote the class of such modules. In case R is coherent, R-mod coincides with the class of finitely presented R-modules.

Proposition 3.6. *The following are equivalent for an R-module M* :

(1) *${}_RM$ is divisible;*
(2) $\mathrm{rad}({}_RM) = M$ (*i.e. M has no maximal R-submodules*)*;*
(3) $\mathfrak{m}M = M$ *for every* $\mathfrak{m} \in \mathrm{Max}(R)$.

Proof. The result is obvious for $M = 0$. So, assume $M \neq 0$. The equivalence (1) $\Leftrightarrow$ (3) is already known for APD's (e.g. L. Salce [47, Proposition 8.1]).

(1) $\Rightarrow$ (2) Suppose that M contains a maximal R-submodule L. Then $M/L \simeq R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \trianglelefteq R$. Since ${}_RM$ is divisible by assumption, it follows that $R/\mathfrak{m}$ is also a divisible R-module (a contradiction).

(2) $\Rightarrow$ (1) Suppose ${}_RM$ is not divisible. Then there exists $0 \neq r \in R$ such that $rM \neq M$. By Lemma 2.2 (5), the non-zero R/rR-module M/rM contains a maximal submodule N/rM. Then there exists $\mathfrak{m} \in \mathrm{Max}(R)$, such that

$$R/\mathfrak{m} \simeq (R/rR)/(\mathfrak{m}/rR) \simeq (M/rM)/(N/rM) \simeq M/N.$$

This implies that $N \in \mathrm{Max}({}_RM)$ (a contradiction).$\blacksquare$

Definition 3.7. A non-empty set $\mathcal{L}$ of R-ideals is said to be a **localizing system** (or a **Gabriel topology**), iff for any ideals $I, J \trianglelefteq R$ we have:
(LS1) If $I \in \mathcal{L}$ and $I \subseteq J$, then $J \in \mathcal{L}$;
(LS2) If $I \in \mathcal{L}$ and $(J :_R r) \in \mathcal{L}$ for every $r \in I$, then $J \in \mathcal{L}$.

Definition 3.8. Let R be an integral domain and $\mathcal{E}$ be a class of R-ideals. We say an R-module M is $\mathcal{E}$-divisible, iff $IM = M$ for every $I \in \mathcal{E}$.

For any classes $\mathcal{M}$ of R-modules and $\mathcal{E}$ of R-ideals we set

$$\begin{array}{lll}\mathcal{D}(\mathcal{M}) & := & \{I \trianglelefteq R \mid IM = M \text{ for every } M \in \mathcal{M}\}; \\ \mathcal{E}\text{-Div} & := & \{{}_RM \mid IM = M \text{ for every } I \in \mathcal{E}\}.\end{array}$$

If R is a domain, then $\mathcal{D}({}_RM)$ is a localizing system by [44, Lemma 1.1].

Lemma 3.9. *Let R be an APD and $\mathfrak{F}$ a localizing system. An R-module M is $\mathfrak{F}$-divisible if and only if $\mathfrak{m}M = M$ for all maximal ideals $\mathfrak{m}$ in $\mathfrak{F}$, i.e.*

$$\mathfrak{F}\text{-Div} = (\mathfrak{F} \cap \mathrm{Max}(R))\text{-Div}. \tag{6}$$

Proof. Let $M \in (\mathfrak{F} \cap \mathrm{Max}(R))$-Div. Let $I \in \mathfrak{F}$ be arbitrary and set $\mathcal{M}(I) := \{\mathfrak{m} \in \mathrm{Max}(R) \mid I \subseteq \mathfrak{m}\} \subseteq \mathfrak{F}$ by (LS1). Let $\mathfrak{m} \in \mathrm{Max}(R)$ be arbitrary. If $\mathfrak{m} \in \mathcal{M}(I)$, then $\mathfrak{m}_{\mathfrak{m}} M_{\mathfrak{m}} = (\mathfrak{m}M)_{\mathfrak{m}} = M_{\mathfrak{m}}$ whence the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is divisible by Proposition 3.6, and it follows that $(IM)_{\mathfrak{m}} = I_{\mathfrak{m}} M_{\mathfrak{m}} = M_{\mathfrak{m}}$. On the other hand, if $\mathfrak{m} \notin \mathcal{M}(I)$, then $I_{\mathfrak{m}} = R_{\mathfrak{m}}$ and so $(IM)_{\mathfrak{m}} = R_{\mathfrak{m}} M_{\mathfrak{m}} = M_{\mathfrak{m}}$. Since $(IM)_{\mathfrak{m}} = M_{\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$, we conclude that $IM = M$ (i.e. $M \in \mathfrak{F}$-Div).■

4. Tilting and Cotilting Modules

This section is devoted to the classification of (co)tilting modules over APD's. For any unexplained definitions we refer to [31].

For any class of R-modules $\mathcal{M}$ we set

$$\begin{array}{lll}\mathcal{M}^{\perp_\infty} & := & \{{}_RN \mid \mathrm{Ext}^i_R(M,N) = 0 \text{ for all } i \geq 1 \text{ and every } M \in \mathcal{M}\}; \\ {}^{\perp_\infty}\mathcal{M} & := & \{{}_RN \mid \mathrm{Ext}^i_R(N,M) = 0 \text{ for all } i \geq 1 \text{ and every } M \in \mathcal{M}\};\end{array}$$

Moreover, we set

$$\mathcal{M}^{\perp} := \bigcap_{M \in \mathcal{M}} \mathrm{Ker}(\mathrm{Ext}^1_R(M,-)) \text{ and } {}^{\perp}\mathcal{M} := \bigcap_{M \in \mathcal{M}} \mathrm{Ker}(\mathrm{Ext}^R_1(-,M)).$$

4.1. For ${}_RX$, let $\mathrm{Gen}_n({}_RX)$ be the class of R-modules M possessing an exact sequence of R-modules $X^{(\Lambda_n)} \to \cdots \to X^{(\Lambda_1)} \to M \to 0$ (for index sets $\Lambda_1, \cdots, \Lambda_n$). Dually, let $\mathrm{Cogen}_n({}_RX)$ be the class of R-modules M possessing an exact sequence of R-modules $0 \to M \to X^{\Lambda_1} \to \cdots \to X^{\Lambda_n}$ (for index sets $\Lambda_1, \cdots, \Lambda_n$). In particular, $\mathrm{Gen}({}_RX) := \mathrm{Gen}_1({}_RX)$ is the class of X-*generated* R-modules and $\mathrm{Cogen}({}_RX) := \mathrm{Cogen}_1({}_RX)$ is the class of X-*cogenerated* R-modules.

4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of R-modules. Then $(\mathcal{A}, \mathcal{B})$ is said to be a **cotorsion pair**, iff $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. If, moreover, $\mathrm{Ext}^i_R(A,B) = 0$ for all $i \geq 1$ and $A \in \mathcal{A}$, $B \in \mathcal{B}$ we say $(\mathcal{A}, \mathcal{B})$ is **hereditary**. Each class $\mathcal{M}$ of R-modules *generates* a cotorsion pair $({}^{\perp}(\mathcal{M}^{\perp}), \mathcal{M}^{\perp})$ and *cogenerates* a cotorsion pair $({}^{\perp}\mathcal{M}, ({}^{\perp}\mathcal{M})^{\perp})$. For two cotorsion pairs $(\mathcal{A}, \mathcal{B})$, $(\mathcal{A}', \mathcal{B}')$, we have $\mathcal{A} = \mathcal{A}'$ if and only if $\mathcal{B} = \mathcal{B}'$.

4.3. An R-module T is said to be n**-tilting**, iff $\mathrm{Gen}_n({}_RT) = T^{\perp_\infty}$; the **induced** n**-tilting class** $T^{\perp_\infty}$ cogenerates a *hereditary cotorsion pair* $({}^{\perp}(T^{\perp_\infty}), T^{\perp_\infty})$ with $\mathcal{A} := {}^{\perp}(T^{\perp_\infty}) \subseteq \mathcal{P}_n$ by [31, Lemma 5.1.8] (in particular, $\mathrm{p.d.}_R(T) \leq n$). By [31, Lemma 6.1.2] (see also [9, Theorem 3.11]), ${}_RT$ is 1-tilting if and only if $\mathrm{Gen}({}_RT) = T^{\perp}$. An R-module T is **tilting**, iff T is n-tilting for some $n \geq 0$. Two tilting R-modules T_1, T_2 are said to be **equivalent** $(T_1 \sim T_2)$, iff $T_1^{\perp_\infty} = T_2^{\perp_\infty}$.

4.4. An R-module C is said to be n**-cotilting**, iff $\mathrm{Cogen}_n({}_RC) = {}^{\perp_\infty}C$; the **induced** n**-cotilting class** ${}^{\perp_\infty}C$ generates a *hereditary cotorsion pair* $({}^{\perp_\infty}C, ({}^{\perp_\infty}C)^{\perp})$ with $\mathcal{B} := ({}^{\perp_\infty}C)^{\perp} \subseteq \mathcal{I}_n$ by [31, Lemma 8.1.4] (in particular, $\mathrm{i.d.}_R(C) \leq n$). By

[31, Lemma 8.2.2] (see also [9, Theorem 3.11]), ${}_RC$ is 1-cotilting if and only if $\mathrm{Cogen}({}_RC) = {}^{\perp}C$. An R-module C is said to be **cotilting**, iff C is n-cotilting for some $n \geq 0$. Two cotilting R-modules C_1, C_2 are said to be **equivalent** ($C_1 \sim C_2$), iff ${}^{\perp_\infty}C_1 = {}^{\perp_\infty}C_2$.

Remark 4.5. Obviously, the 0-tilting modules are precisely the projective generators, while the 0-cotilting modules are precisely the injective cogenerators.

Example 4.6. Let R be an integral domain, $S \subseteq R^\times$ a multiplicative subset, and $\omega = ()$ be the empty sequence. Let F be the *free* R-module with basis

$$\beta := \{(s_0, \cdots, s_n) \mid n \geq 0 \text{ and } s_j \in S \text{ for } 0 \leq j \leq n\} \cup \{\omega\}$$

and G the R-submodule of F (which is in fact *free*) generated by

$$\{(s_0, \cdots, s_n)s_n - (s_0, \cdots, s_{n-1}) \mid n > 0 \text{ and } s_j \in S \text{ for } 0 \leq j \leq n\} \cup \{(s)s - \omega\}.$$

The R-module $\delta_S := F/G$ is a 1-tilting R-module with $\delta_S^{\perp} = \mathrm{Gen}(\delta_S) = \mathcal{D}_S$ as shown in [27] and we call it the **Fuchs-Salce module.** It generalizes the **Fuchs module** $\delta := \delta_{R^\times}$ (introduced in [29]), which was studied and shown to be 1-tilting with $\delta^{\perp} = \mathrm{Gen}({}_R\delta) = \mathcal{DI}$ by A. Facchini in [24] and [25].

Definition 4.7. ([31]) A **Matlis localization** of the commutative ring R is $S^{-1}R$, where $S \subseteq R^\times$ is a multiplicative subset and $\mathrm{p.d.}_R(S^{-1}R) \leq 1$.

Lemma 4.8. ([31, Proposition 5.2.24], [3, Theorem 1.1]) *Let R be a commutative ring and $S \subseteq R^\times$ a multiplicative subset.*

(1) Let T be an n-tilting R-module, $\mathcal{T} := T^{\perp_\infty}$ the induced n-tilting class and

$$\mathcal{T}_S := \{{}_{S^{-1}R}N \mid N \simeq S^{-1}M \text{ for some } M \in \mathcal{T}\}.$$

Then $S^{-1}T$ is an n-tilting $S^{-1}R$-module and its induced n-tilting class is

$$(S^{-1}T)^{\perp_\infty} := \bigcap_{i \geq 1} \mathrm{Ker}(\mathrm{Ext}^i_{S^{-1}R}(S^{-1}T, -)) = \mathcal{T}_S = T^{\perp_\infty} \cap S^{-1}R\text{-Mod}.$$

Moreover, ${}_RM \in \mathcal{T}$ if and only if $M_{\mathfrak{m}} \in \mathcal{T}_{\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$. If T' is another n-tilting R-module, then

$$T \sim T' \Leftrightarrow T_{\mathfrak{m}} \sim T'_{\mathfrak{m}} \text{ for all maximal ideals } \mathfrak{m} \in \mathrm{Max}(R). \tag{7}$$

(2) The following are equivalent:

(a) $\mathrm{p.d.}_R(S^{-1}R) \leq 1$ *(i.e. $S^{-1}R$ is a Matlis localization);*

(b) $T(S) := S^{-1}R \oplus \frac{S^{-1}R}{R}$ *is a 1-tilting R-module;*

(c) $\mathrm{Gen}({}_RS^{-1}R) = \mathcal{D}_S$.

Moreover, in this case $T(S)^{\perp_\infty} = \mathrm{Gen}(T(S)) = \mathcal{D}_S$.

We prove now some fundamental properties of (co)tilting modules over APD's, some of which are analogous to the case of Prüfer domains:

Proposition 4.9. *Let R be an APD with $R \neq Q$.*

(1) All tilting R-modules are 1-tilting.

(2) The torsion-free tilting R-modules are precisely the projective generators (i.e. the 0-tilting R-modules) and are all equivalent to R.

(3) Every divisible tilting R-modules generates $\mathcal{DI}$, whence is equivalent to δ.

(4) All localizations of R are Matlis localizations. For every multiplicative subset $S \subseteq R^\times$ we have a tilting R-module $T(S) := S^{-1}R \oplus S^{-1}R/R \sim \delta_S$ and a cotilting R-module $T(S)^c \sim \delta_S^c$.

(5) All cotilting R-modules are 1-cotilting.

(6) The divisible cotilting R-modules are precisely the injective cogenerators (i.e. the 0-cotilting R-modules) and are equivalent to $R^c := \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$.

Proof. (1) Follows directly from $\mathcal{P} = \mathcal{P}_1$ (3).

(2) If ${}_RT$ is a torsion-free tilting R-module, then by "1": $T \in \mathcal{TF} \cap \mathcal{P}_1 \overset{(5)}{=} \mathcal{FL}$, whence ${}_RT$ is projective (since flat 1-tilting modules over arbitrary rings are projective by [11, Corollary 2.8]). In this case, $\mathrm{Gen}({}_RT) = T^{\perp} = R$-Mod $= R^{\perp}$; consequently, ${}_RT$ is a projective generator and $T \sim R$.

(3) Recall that $\mathcal{F}_1$ generates a cotorsion pair $(\mathcal{F}_1, \mathcal{WI})$, where (by definition) $\mathcal{WI} := \mathcal{F}_1^{\perp}$ is the class of *weak-injective* R-modules. Notice that conditions (8) and (9) of Lemma 2.6 can be expressed as $(\mathcal{F}_1, \mathcal{WI}) = (\mathcal{P}_1, \mathcal{DI})$. Let T be a tilting R-module and consider the induced cotorsion pair $({}^{\perp}(T^{\perp}), T^{\perp})$. If ${}_RT$ is divisible, then $T^{\perp} = \mathrm{Gen}({}_RT) \subseteq \mathcal{DI}$, whence $\mathcal{P}_1 = {}^{\perp}\mathcal{DI} \subseteq {}^{\perp}(T^{\perp}) \subseteq \mathcal{P}_1$. So, $\delta^{\perp} = \mathcal{DI} = \mathcal{P}_1^{\perp} = T^{\perp} = \mathrm{Gen}({}_RT)$, i.e. T generates $\mathcal{DI}$ and $T \sim \delta$.

(4) For every multiplicative subset $S \subseteq R^{\times}$, the localization $S^{-1}R$ is a flat R-module whence $\mathrm{p.d.}_R(S^{-1}R) \leq 1$ by Lemma 2.6 (9). It follows by Lemma 4.8 (2) that $T(S) := S^{-1}R \oplus \frac{S^{-1}R}{R}$ is a tilting R-module with $T(S)^{\perp} = \mathcal{D}_S = \delta_S^{\perp}$, whence $T(S) \sim \delta_S$. The character module of any tilting R-module is cotilting by [31, Theorem 8.1.2], whence $T(S)^c$ is a cotilting R-module which is equivalent to δ_S^c (e.g. [31, Theorem 8.1.13]).

(5) Follows directly from $\mathcal{I} = \mathcal{I}_1$ (3).

(6) If ${}_RC$ is a divisible cotilting R-module, then by "6": $C \in \mathcal{DI} \cap \mathcal{I}_1 \overset{(2)}{=} \mathcal{IN}$. In this case, $\mathrm{Cogen}({}_RC) = {}^{\perp}C = R$-Mod $= {}^{\perp}R^c$; consequently, ${}_RC$ is an injective cogenerator and $C \sim R^c$.■

The following is a key-result that will be used frequently in the sequel.

Theorem 4.10. *Let $(R, \mathfrak{m})$ be a local APD with $R \neq Q$. Any tilting R-module is either projective or divisible. Hence, R has exactly two tilting modules $\{R, \delta\}$ (up to equivalence) and exactly two tilting classes $\{R$-Mod, $\mathcal{DI}\}$.*

Proof. Let T be a tilting R-module and assume that ${}_RT$ is not divisible. Then $T \neq 0$ and contains by Proposition 3.6 a maximal R-submodule N such that $T/N \simeq R/\mathfrak{m}$. By [15] all tilting modules (over arbitrary rings) are of finite type. So, there exists $\mathcal{S} \subseteq \mathcal{P}_1 \cap R$-mod such that $R/\mathfrak{m} \in \mathrm{Gen}({}_RT) = T^{\perp} = \mathcal{S}^{\perp}$. Let $M \in \mathcal{S}$ be arbitrary, so that $\mathrm{Ext}_R^1(M, R/\mathfrak{m}) = 0$. Since the field $R/\mathfrak{m}$ is indeed injective as a module over itself, it follows (e.g. [28, Page 34 (6)]) that

$$\begin{aligned}\mathrm{Tor}_1^R(R/\mathfrak{m}, M) &\simeq \mathrm{Tor}_1^R(\mathrm{Hom}_{R/\mathfrak{m}}(R/\mathfrak{m}, R/\mathfrak{m}), M) \\ &\simeq \mathrm{Hom}_{R/\mathfrak{m}}(\mathrm{Ext}_R^1(M, R/\mathfrak{m}), R/\mathfrak{m}) = 0.\end{aligned}$$

By [12, II.3.2.Corollary 2], ${}_RM$ is projective (being finitely presented and flat). So, $\mathcal{S} \subseteq \mathcal{PR}$, whence ${}_RT$ is projective.■

Recall (from [32]) that an R-submodule M of an R-module N is said to be a **restriction submodule**, iff $M_{\mathfrak{m}} = N_{\mathfrak{m}}$ or $M_{\mathfrak{m}} = 0$ for every $\mathfrak{m} \in \mathrm{Max}(R)$. For any subset $X \subseteq \mathrm{Max}(R)$, we set

$$R_{(X)} := \bigcap_{\mathfrak{m} \in X} R_{\mathfrak{m}} \ (:= Q, \text{ if } X = \varnothing) \ .$$

Lemma 4.11. *Let $R \neq Q$, $X \subseteq \mathrm{Max}(R)$, $X' := \mathrm{Max}(R)\backslash X$ and consider*

$$M_1 := \frac{R_{(X)}}{R} \text{ and } M_2 := \frac{R_{(X')}}{R}.$$

(1) If R is an h-local domain, then $M_1, M_2 \subseteq \frac{Q}{R}$ are restriction R-submodules and

$$\frac{Q}{R} = M_1 \oplus M_2 = \frac{R_{(X)}}{R} \oplus \frac{R_{(X')}}{R}. \tag{8}$$

(2) If R is a 1-dimensional h-local domain, then

$$T(X) := R_{(X)} \bigoplus \frac{R_{(X)}}{R} \quad (= Q \oplus \frac{Q}{R},\ \text{if } X = \varnothing)$$

is a 1-tilting R-module.

Proof. Recall first that if $\mathfrak{m}, \mathfrak{m}' \in \mathrm{Max}(R)$ are such that $\mathfrak{m} \neq \mathfrak{m}'$, then we have by [37, Theorem 3.19] (see also [28, IV.3.2]):

$$R_{\mathfrak{m}} \otimes_R R_{\mathfrak{m}'} \simeq (R_{\mathfrak{m}})_{\mathfrak{m}'} = Q. \tag{9}$$

Moreover, if $\{R_\lambda\}_\Lambda$ is a class of R-submodules of Q with $\bigcap_{\lambda\in\Lambda} R_\lambda \neq 0$, then it follows from [28, IV.3.10] that

$$(\bigcap_{\lambda\in\Lambda} R_\lambda)_{\mathfrak{m}} = \bigcap_{\lambda\in\Lambda} (R_\lambda)_{\mathfrak{m}} \text{ for every } \mathfrak{m} \in \mathrm{Max}(R). \tag{10}$$

(1) Clearly $M_1 \cap M_2 = 0$. Let $\mathfrak{m}' \in \mathrm{Max}(R)$ be arbitrary. Then

$$(M_1)_{\mathfrak{m}'} = \frac{(R_{(X)})_{\mathfrak{m}'}}{R_{\mathfrak{m}'}} \overset{(10)}{=} \frac{\bigcap_{\mathfrak{m}\in X} (R_{\mathfrak{m}})_{\mathfrak{m}'}}{R_{\mathfrak{m}'}} \overset{(9)}{=} \begin{cases} 0, & \mathfrak{m}' \in X \\ \frac{Q}{R_{\mathfrak{m}'}}, & \mathfrak{m}' \notin X \end{cases}.$$

Similarly,

$$(M_2)_{\mathfrak{m}'} = \begin{cases} \frac{Q}{R_{\mathfrak{m}'}}, & \mathfrak{m}' \in X \\ 0, & \mathfrak{m}' \notin X \end{cases}.$$

So, $M_1, M_2 \subseteq \frac{Q}{R}$ are restriction R-submodules. Moreover, we have $(M_1 \oplus M_2)_{\mathfrak{m}'} = (M_1)_{\mathfrak{m}'} \oplus (M_2)_{\mathfrak{m}'} = \frac{Q}{R_{\mathfrak{m}'}} = (\frac{Q}{R})_{\mathfrak{m}'}$ for all $\mathfrak{m}' \in \mathrm{Max}(R)$, and so $\frac{Q}{R} = M_1 \oplus M_2$.

(2) Notice first that a 1-dimensional h-local domain is a Matlis domain (in fact $\mathrm{p.d.}_R(Q) = \mathrm{p.d.}_R(\frac{Q}{R}) = 1$ as shown in [47, Lemma 2.4]). For any $X \subseteq \mathrm{Max}(R)$, we have $\frac{Q}{R} \overset{(8)}{=} \frac{R_{(X)}}{R} \oplus \frac{R_{(X')}}{R}$ and so $T(X)$ is a 1-tilting R-module by [3, Theorem 8.2].■

Remark 4.12. Although we proved (8) for general h-local domains, we point out here that it can be obtained for an *APD* R by applying [3, Theorem 3.10] to $M_1 := \frac{R_{(X)}}{R}$. Then $X_1 := \mathrm{Supp}(M_1) = \mathrm{Max}(R)\backslash X$ and $X_2 := \mathrm{Supp}(Q/R)\backslash X_1 = X$. Consider the embedding $\varphi : \frac{Q}{R} \to \prod_{\mathfrak{m}\in\mathrm{Max}(R)} (\frac{Q}{R})_{\mathfrak{m}}$. Since R is h-local, it follows by [28, Theorem IV.3.7] (3) that $M_1 \simeq \bigoplus_{\mathfrak{m}\notin\mathrm{Max}(R)} (M_1)_{\mathfrak{m}} = \bigoplus_{\mathfrak{m}\in X} \frac{Q}{R_{\mathfrak{m}}}$. So, $M_2 := \varphi^{-1}(\prod_{\mathfrak{m}\in X} (\frac{Q}{R})_{\mathfrak{m}}) = \frac{R_{(X')}}{R}$. Notice that $\mathrm{w.d.}_R(\frac{Q}{R_{(X)}}) \leq 1$ and so $\mathrm{p.d.}_R(\frac{Q}{R_{(X)}}) \leq 1$ by Lemma 2.6 (9). The equality (8) follows now by [3, Theorem 3.10].

Lemma 4.13. *Let R be an APD with $R \neq Q$. If T is a tilting R-module, then*

$$T^{\perp_\infty} = \mathrm{Gen}({}_RT) = \mathcal{D}({}_RT)\text{-Div}. \tag{11}$$

Proof. Clearly $\mathrm{Gen}({}_RT) \subseteq \mathcal{D}(T)$-Div. Let $M \in \mathcal{D}(T)$-Div, $\mathfrak{m} \in \mathrm{Max}(R)$ be arbitrary and consider the tilting $R_\mathfrak{m}$-module $T_\mathfrak{m}$. By Theorem 4.10, ${}_{R_\mathfrak{m}}T_\mathfrak{m}$ is either divisible or projective. If $\mathfrak{m} \in \mathcal{D}(T)$, then $T_\mathfrak{m}$ is divisible and generates all divisible $R_\mathfrak{m}$-modules by Proposition 4.9 (3). Moreover, $\mathfrak{m}_\mathfrak{m}M_\mathfrak{m} = (\mathfrak{m}M)_\mathfrak{m} = M_\mathfrak{m}$ and it follows by Proposition 3.6 that $M_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module, whence $M_\mathfrak{m} \in \mathrm{Gen}({}_{R_\mathfrak{m}}T_\mathfrak{m})$. On the other hand, if $\mathfrak{m} \notin \mathcal{D}(T)$ then $T_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module whence a generator in $R_\mathfrak{m}$-Mod by Proposition 4.9 (2). In either cases $M_\mathfrak{m} \in \mathrm{Gen}({}_{R_\mathfrak{m}}T_\mathfrak{m}) = T_\mathfrak{m}^{\perp_\infty}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$, whence $M \in T^{\perp_\infty} = \mathrm{Gen}({}_RT)$ by Lemma 4.8 (1).■

Theorem 4.14. *Let R be an APD with $R \neq Q$.*

(1) *The set*

$$\{T(X) \mid X \subseteq \mathrm{Max}(R)\}$$

is a representative set (up to equivalence) of all tilting R-modules.

(2) *There is a bijective correspondence between the set of all tilting torsion classes of R-modules and the power set of the maximal spectrum $\mathfrak{B}(\mathrm{Max}(R))$. The correspondence is given by the mutually inverse assignments:*

$$\mathcal{T} \mapsto \quad \mathcal{DM}(\mathcal{T}) \quad := \quad \{\mathfrak{m} \in \mathrm{Max}(R) \mid \mathfrak{m}M = M \textit{ for every } M \in \mathcal{T}\};$$

and

$$X \mapsto \quad X\text{-Div} \quad := \quad \{{}_RM \mid \mathfrak{m}M = M \textit{ for every } \mathfrak{m} \in X\}.$$

(3) *If R is coprimely packed, then the class of Fuchs-Salce tilting modules*

$$\{\delta_S \mid S \subseteq R^\times \textit{ is a multiplicative subset}\}$$

classifies all tilting R-modules (up to equivalence).

Proof. (1) Let T be a tilting R-module and set

$$\begin{aligned} \Omega_1 &:= \{\mathfrak{m} \in \mathrm{Max}(R) \mid T_\mathfrak{m} \text{ is a divisible } R_\mathfrak{m}\text{-module}\}; \\ \Omega_2 &:= \{\mathfrak{m} \in \mathrm{Max}(R) \mid T_\mathfrak{m} \text{ is a projective } R_\mathfrak{m}\text{-module}\}. \end{aligned}$$

Notice first that $\mathrm{Max}(R) = \Omega_1 \cup \Omega_2$ by Theorem 4.10 (a disjoint union by applying Proposition 4.9 (2) & (3) to the ring $R_\mathfrak{m}$).

Claim: $T \sim T(\Omega_2)$. One can show (as in the proof of Lemma 4.11), that if $\mathfrak{m} \in \mathrm{Max}(R)$ then

$$T(\Omega_2)_\mathfrak{m} = \begin{cases} Q \oplus \frac{Q}{R_\mathfrak{m}}, & \mathfrak{m} \in \Omega_1 \\ \\ R_\mathfrak{m}, & \mathfrak{m} \in \Omega_2 \end{cases}.$$

So, $T_\mathfrak{m} \sim T(\Omega_2)_\mathfrak{m}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$ whence $T \sim T(\Omega_2)$ by (7).

(2) Let $\mathcal{T} = T^{\perp_\infty}$ be a tilting torsion class for some tilting R-module T. Then

$$\mathcal{DM}(\mathcal{T})\text{-Div} = \mathcal{DM}(T)\text{-Div} \overset{(6)}{=} \mathcal{D}(T)\text{-Div} \overset{(11)}{=} \mathrm{Gen}({}_RT) = T^{\perp_\infty} = \mathcal{T}.$$

On the other hand, let $X \subseteq \mathrm{Max}(R)$, $\overline{X} := \mathrm{Max}(R)\backslash X$, and $T' := T(\overline{X})$. Then clearly $\mathcal{DM}(T') = X$ and so

$$\mathcal{DM}(X\text{-Div}) = \mathcal{DM}(\mathcal{DM}(T')\text{-Div}) = \mathcal{DM}(T') = X.$$

(3) Let R be compactly packed. Let Ω_1 and Ω_2 be as in "1".

Case 1. $\mathrm{Max}(R) = \Omega_1$ (i.e. $T_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \mathrm{Max}(R)$). In this case, ${}_RT$ is divisible whence $T \sim Q \oplus Q/R$ and we can take $S = R^\times$.

Case 2. $\mathrm{Max}(R) = \Omega_2$ (i.e. $T_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \mathrm{Max}(R)$). In this case, ${}_RT$ is projective whence $T \sim R$ and we can take $S = \{1\}$.

Case 3. $\mathrm{Max}(R) \neq \Omega_1$ and $\mathrm{Max}(R) \neq \Omega_2$. Let

$$S := R \backslash \bigcup_{\mathfrak{m} \in \Omega_2} \mathfrak{m} \text{ and } T(S) := S^{-1}R \oplus S^{-1}R/R.$$

Let $\mathfrak{m} \in \Omega_2$, so that $T_\mathfrak{m}$ is projective and $S \subseteq R\backslash\mathfrak{m}$. Then $(S^{-1}R)_\mathfrak{m} = R_\mathfrak{m}$. Therefore $(T(S))_\mathfrak{m} = (S^{-1}R)_\mathfrak{m} \oplus (S^{-1}R/R)_\mathfrak{m} = R_\mathfrak{m}$ is equivalent to the projective $R_\mathfrak{m}$-module $T_\mathfrak{m}$. On the other hand, let $\mathfrak{m} \in \Omega_1$ so that $T_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module. Then $\mathfrak{m} \cap S \neq \varnothing$ (otherwise $\mathfrak{m} \subseteq \bigcup\limits_{\mathfrak{m} \in \Omega_2} \mathfrak{m}$ and so $\mathfrak{m} \in \Omega_2$ since R is coprimely packed; a contradiction since $\Omega_1 \cap \Omega_2 = \varnothing$). Let $\widetilde{s} \in S \cap \mathfrak{m}$. Clearly $\widetilde{s}(S^{-1}R)_\mathfrak{m} = (S^{-1}R)_\mathfrak{m}$, whence $(S^{-1}R)_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module by Proposition 3.6. It follows that $(T(S))_\mathfrak{m} = (S^{-1}R)_\mathfrak{m} \oplus (S^{-1}R)_\mathfrak{m}/R_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module, whence $T(S)_\mathfrak{m} \sim T_\mathfrak{m}$ as $R_\mathfrak{m}$-modules by Proposition 4.9 (3) (applied to the ring $R_\mathfrak{m}$). Since $T_\mathfrak{m} \sim T(S)_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(R)$, we conclude that $T \sim T(S)$ by (7).$\blacksquare$

Remark 4.15. Let R be a 1-Gorenstein ring and ${}_RT$ be a tilting R-module. By [52] there exists $X \subseteq \mathbf{P}_1$ (the set of prime ideals of height 1) and some (*unique*) R-module R_X, satisfying $R \subseteq R_X \subseteq Q$ and fitting in an exact sequence

$$0 \to R \to R_X \to \bigoplus_{\mathfrak{m} \in X} \mathrm{E}(R/\mathfrak{m}) \to 0,$$

such that T is equivalent to the so-called **Bass tilting module** $B(X) := R_X \oplus \bigoplus_{\mathfrak{m} \in X} \mathrm{E}(R/\mathfrak{m})$. Let $\mathfrak{m} \in \mathrm{Max}(R)$ be arbitrary. By the proof of [52, Theorem 0.1], the $R_\mathfrak{m}$-module $B(X)_\mathfrak{m}$ is injective, whence divisible, if $\mathfrak{m} \in X$ and projective if $\mathfrak{m} \notin X$. If R is a 1-Gorenstein domain (whence an APD), the same holds for the $R_\mathfrak{m}$-module $T(X')_\mathfrak{m}$, where $X' := \mathrm{Max}(R)\backslash X$. It follows that, in this case, $B(X) \sim T(X')$ by (7) and so $T \sim T(X')$.$\blacksquare$

A direct application of Theorem 4.14, and [31, Theorem 8.2.8] yields

Corollary 4.16. *Let R be a coherent* (*Noetherian*) *APD.*

(1) *All cotilting R-modules are of cofinite type and $\{T(X)^c \mid X \subseteq \mathrm{Max}(R)\}$ is a representative set* (*up to equivalence*) *of all cotilting R-modules.*
(2) *If R is coprimely packed, then $\{\delta_S^c \mid S \subseteq R^\times$ is a multiplicative subset$\}$ classifies all cotilting R-modules* (*up to equivalence*).

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